Nine classes of integrable boundary conditions for the eight-state supersymmetric fermion model

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Nine classes of integrable boundary conditions for the eight-state supersymmetric model of strongly correlated fermions are presented. The boundary systems are solved by using the coordinate Bethe ansatz method and the Bethe ansatz equations for all nine cases are given.

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I. INTRODUCTION

Lattice integrable models with open-boundary conditions are one of the recent developments which deserves careful elaborations. As many systems in nature are confined in finite boxes (or intervals for one-dimensional systems), the effects of boundaries are very significant. This is particularly so for integrable systems since boundary conditions generally spoil the integrability of the bulk models. Therefore, the problem of how to extend a bulk integrable model to include integrable boundary conditions becomes very important.

A systematic method for treating integrable lattice models with boundaries, that is the boundary quantum inverse scattering method (QISM), has been developed by Sklyanin [1] and generalized in [2,3]. Within this framework, the integrable boundary conditions are determined by boundary K-matrices obeying the (graded) reflection equations.

Integrable correlated fermion systems constitute an important class of lattice integrable models, which have recently attracted much attention [4–8]. In [8], we proposed two new integrable models with Lie superalgebra $gl(3|1)$ and quantum superalgebra $U_q[gl(3|1)]$ symmetries, respectively. These are eight-state fermion models with correlated single-particle and pair hoppings, uncorrelated triple-particle hopping and two- and three-particle on-site interactions. By eight-state, we mean that at a given lattice site j of the length L there are eight possible states:

$$\begin{aligned}
&|0\rangle\,, \quad c_{j,+}^\dagger|0\rangle\,, \quad c_{j,0}^\dagger|0\rangle\,, \quad c_{j,-}^\dagger|0\rangle,\\
&c_{j,+}^\dagger c_{j,0}^\dagger|0\rangle\,, \quad c_{j,+}^\dagger c_{j,-}^\dagger|0\rangle\,, \quad c_{j,0}^\dagger c_{j,-}^\dagger|0\rangle\,, \quad c_{j,+}^\dagger c_{j,0}^\dagger c_{j,-}^\dagger|0\rangle\,,
\end{aligned} \tag{I.1}$$

where $c_{j,\alpha}^\dagger$ ($c_{j,\alpha}$) denotes a fermionic creation (annihilation) operator which creates (annihilates) a fermion of species $\alpha = +,\ 0,\ -$ at site j; these operators satisfy the anti-commutation relations given by $\{c_{i,\alpha}^\dagger, c_{j,\beta}\} = \delta_{ij}\delta_{\alpha\beta}$.

In a series of papers, we have constructed a large number of integrable boundary conditions for various models of strongly correlated electrons [9,10,3]. In this paper, we are concerned with the integrable eight-state fermion model with Lie superalgebra $gl(3|1)$ symmetry. We present nine classes of boundary conditions for this model, all of which are shown to be integrable by the graded boundary QISM recently formulated in [3]. We solve the boundary systems by using the coordinate Bethe ansatz method and derive the Bethe ansatz equations for all nine cases.

This paper is organized as follows. In Section II the boundary model Hamiltonians are described. In the following sections we establish the quantum integrability for all these boundary conditions,and derive the corresponding Bethe ansatz equations in terms of the coordinate Bethe ansatz method. The last section is devoted to the conclusion.

II. BOUNDARY MODEL HAMILTONIANS

We consider the following Hamiltonian with boundary terms

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$$H=\sum_{j=1}^{L-1} H^{\rm bulk}_{j,j+1}+H^{\rm boundary}_{lt}+H^{\rm boundary}_{rt}, \tag{II.1}$$

where $H^{\rm boundary}_{lt}$, $H^{\rm boundary}_{rt}$ are respectively left and right boundary terms whose explicit forms are given below, and $H^{\rm bulk}_{j,j+1}$ is the Hamiltonian density of the eight-state supersymmetric U model [8]

$$\begin{aligned}
H^{\rm bulk}_{j,j+1}(g)=&-\sum_{\alpha}(c^\dagger_{j,\alpha}c_{j+1,\alpha}+{\rm h.c.})\ \exp\left\{-\frac{\eta}{2}\sum_{\beta(\neq\alpha)}(n_{j,\beta}+n_{j+1,\beta})+\frac{\zeta}{2}\sum_{\beta\neq\gamma(\neq\alpha)}(n_{j,\beta}n_{j,\gamma}+n_{j+1,\beta}n_{j+1,\gamma})\right\}\\
&-\frac{1}{2(g+1)}\sum_{\alpha\neq\beta\neq\gamma}(c^\dagger_{j,\alpha}c^\dagger_{j,\beta}c_{j+1,\beta}c_{j+1,\alpha}+{\rm h.c.})\ \exp\left\{-\frac{\xi}{2}(n_{j,\gamma}+n_{j+1,\gamma})\right\}\\
&-\frac{2}{(g+1)(g+2)}\left(c^\dagger_{j,+}c^\dagger_{j,0}c^\dagger_{j,-}c_{j+1,-}c_{j+1,0}c_{j+1,+}+{\rm h.c.}\right)\\
&+\sum_{\alpha}(n_{j,\alpha}+n_{j+1,\alpha})-\frac{1}{2(g+1)}\sum_{\alpha\neq\beta}(n_{j,\alpha}n_{j,\beta}+n_{j+1,\alpha}n_{j+1,\beta})\\
&+\frac{2}{(g+1)(g+2)}(n_{j,+}n_{j,0}n_{j,-}+n_{j+1,+}n_{j+1,0}n_{j+1,-}),
\end{aligned}\tag{II.2}$$

where $n_{j\alpha}$ is the number density operator $n_{j\alpha}=c^\dagger_{j\alpha}c_{j\alpha}$, $n_j=n_{j+}+n_{j0}+n_{j-}$; and

$$\eta=-\ln\frac{g}{g+1},\quad \zeta=\ln(g+1)-\frac{1}{2}\ln g(g+2),\quad \xi=-\ln\frac{g}{g+2}. \tag{II.3}$$

We claim that the boundary Hamiltonain (II.1) is integrable under the boundary conditions:

$$\begin{aligned}
\text{Case (i)}:\quad H^{\rm boundary}_{lt}&=-\frac{2g}{2-\xi^I_-}\left(n_1-\frac{2}{\xi^I_-}(n_{1+}n_{10}+n_{10}n_{1-}+n_{1+}n_{1-})+\frac{8}{\xi^I_-(2+\xi^I_-)}n_{1+}n_{10}n_{1-}\right),\\
H^{\rm boundary}_{rt}&=-\frac{2g}{2-\xi^I_+}\left(n_L-\frac{2}{\xi^I_+}(n_{L+}n_{L0}+n_{L0}n_{L-}+n_{L+}n_{L-})+\frac{8}{\xi^I_+(2+\xi^I_+)}n_{L+}n_{L0}n_{L-}\right);
\end{aligned}\tag{II.4}$$

$$\begin{aligned}
\text{Case (ii)}:\quad H^{\rm boundary}_{lt}&=\frac{2g}{\xi^{II}_-}\left(n_{10}+n_{1-}-\frac{2}{2-\xi^{II}_-}n_{10}n_{1-}\right),\\
H^{\rm boundary}_{rt}&=\frac{2g}{\xi^{II}_+}\left(n_{L0}+n_{L-}-\frac{2}{2-\xi^{II}_+}n_{L0}n_{L-}\right);
\end{aligned}\tag{II.5}$$

$$\text{Case (iii)}:\quad H^{\rm boundary}_{lt}=\frac{2g}{\xi^{III}_-}\ n_{1-},\qquad H^{\rm boundary}_{rt}=\frac{2g}{\xi^{III}_+}\ n_{L-}, \tag{II.6}$$

plus six others, numbered from Case (iv) to Case (ix) below, which are built from the above three cases by using the fact that boundary conditions on the left and on the right end of an open lattice chain can be chosen independently. Throughout, $\xi^a_\pm(a=I,II,III)$ are some parameters describing the boundary effects.

III. QUANTUM INTEGRABILITY

Quantum integrability of the boundary conditions proposed in the previous section can be established by means of the (graded) boundary QISM recently formulated in [3]. Indeed, the integrability corresponding to the above Case (i) has been shown in [10]. We now establish the integrability for the remaining eight cases. We first search for boundary K-matrices which satisfy the graded reflection equations:

$$R_{12}(u_1-u_2)\stackrel{1}{K}_-(u_1)R_{21}(u_1+u_2)\stackrel{2}{K}_-(u_2)=\stackrel{2}{K}_-(u_2)R_{12}(u_1+u_2)\stackrel{1}{K}_-(u_1)R_{21}(u_1-u_2), \tag{III.1}$$

$$\begin{aligned}
&R^{st_1ist_2}_{21}(-u_1+u_2)\stackrel{1}{K}{}^{st_1}_+(u_1)R_{12}(-u_1-u_2+4)\stackrel{2}{K}{}^{ist_2}_+(u_2)\\
&\qquad=\stackrel{2}{K}{}^{ist_2}_+(u_2)R_{21}(-u_1-u_2+4)\stackrel{1}{K}{}^{st_1}_+(u_1)R^{st_1ist_2}_{12}(-u_1+u_2),
\end{aligned}\tag{III.2}$$

where $R(u) \in End(V \otimes V)$, with V being 8-dimensional representation of $gl(3|1)$, is the R-matrix of the eight-state supersymmetric U model [8], and $R_{21}(u) = P_{12}R_{12}(u)P_{12}$ with P being the graded permutation operator; the supertransposition st_μ $(\mu = 1, 2)$ is only carried out in the μ-th factor superspace of $V \otimes V$, whereas ist_μ denotes the inverse operation of st_μ.

It can be checked that there are three different diagonal boundary K-matrices [1], $K_-^I(u),\ K_-^{II}(u),\ K_-^{III}(u)$, which solve the first reflection equation (III.1):

$$
\begin{aligned}
K_-^I(u) &= \frac{1}{\xi_-^I(2-\xi_-^I)(2+\xi_-^I)}\mathrm{diag}\left(A_-^I(u), B_-^I(u), B_-^I(u), B_-^I(u), C_-^I(u), C_-^I(u), C_-^I(u), D_-^I(u)\right), \\
K_-^{II}(u) &= \frac{1}{\xi_-^{II}(2-\xi_-^{II})}\mathrm{diag}\left(A_-^{II}(u), A_-^{II}(u), B_-^{II}(u), B_-^{II}(u), B_-^{II}(u), B_-^{II}(u), C_-^{II}(u), C_-^{II}(u)\right), \\
K_-^{III}(u) &= \frac{1}{\xi_-^{III}}\mathrm{diag}\left(A_-^{III}(u), A_-^{III}(u), A_-^{III}(u), B_-^{III}(u), A_-^{III}(u), B_-^{III}(u), B_-^{III}(u), B_-^{III}(u)\right),
\end{aligned}
\tag{III.3}
$$

where

$$
\begin{aligned}
A_-^I(u) &= (-\xi_-^I + u)(2 - \xi_-^I + u)(-2 - \xi_-^I + u), \\
B_-^I(u) &= (-\xi_-^I + u)(2 - \xi_-^I - u)(-2 - \xi_-^I + u), \\
C_-^I(u) &= (-\xi_-^I - u)(2 - \xi_-^I - u)(-2 - \xi_-^I + u), \\
D_-^I(u) &= (-\xi_-^I - u)(2 - \xi_-^I - u)(-2 - \xi_-^I - u), \\
A_-^{II}(u) &= (\xi_-^{II} + u)(2 - \xi_-^{II} - u), \\
B_-^{II}(u) &= (\xi_-^{II} - u)(2 - \xi_-^{II} - u), \\
C_-^{II}(u) &= (\xi_-^{II} - u)(2 - \xi_-^{II} + u), \\
A_-^{III}(u) &= (\xi_-^{III} + u), \qquad B_-^{III}(u) = (\xi_-^{III} - u).
\end{aligned}
\tag{III.4}
$$

The corresponding K-matrices, $K_+^I(u),\ K_+^{II}(u),\ K_+^{III}(u)$, can be obtained from the isomorphism of the two reflection equations. Indeed, given a solution $K_-^a(u)$ of (III.1), then $K_+^a(u)$ defined by

$$
K_+^{a\,st}(u) = K_-^a(-u+2), \qquad a = I,\ II,\ III,
\tag{III.5}
$$

are solutions of (III.2). The proof follows from some algebraic computations upon substituting (III.5) into (III.2) and making use of the properties of the R-matrix .

Following Sklyanin's arguments [1], one may show that the quantity $\mathcal{T}_-(u)$ given by

$$
\mathcal{T}_-(u) = T(u)K_-(u)T^{-1}(-u), \qquad T(u) = R_{0L}(u)\cdots R_{01}(u),
\tag{III.6}
$$

satisfies the same relation as $K_-(u)$:

$$
R_{12}(u_1-u_2)\stackrel{1}{\mathcal{T}}_-(u_1)R_{21}(u_1+u_2)\stackrel{2}{\mathcal{T}}_-(u_2) = \stackrel{2}{\mathcal{T}}_-(u_2)R_{12}(u_1+u_2)\stackrel{1}{\mathcal{T}}_-(u_1)R_{21}(u_1-u_2).
\tag{III.7}
$$

Thus if one defines the boundary transfer matrix $\tau(u)$ as

$$
\tau(u) = str(K_+(u)\mathcal{T}_-(u)) = str\left(K_+(u)T(u)K_-(u)T^{-1}(-u)\right),
\tag{III.8}
$$

then it can be shown [3] that $[\tau(u_1), \tau(u_2)] = 0$. Since $K_\pm(u)$ can be taken as $K_\pm^I(u),\ K_\pm^{II}(u)$ and $K_\pm^{III}(u)$, respectively, we have nine possible choices of boundary transfer matrices:

$$
\tau^{(a,b)}(u) = str\left(K_+^a(u)T(u)K_-^b(u)T^{-1}(-u)\right), \quad a,\ b = I,\ II,\ III,
\tag{III.9}
$$

which reflects the fact that the boundary conditions on the left end and on the right end of the open lattice chain are independent.

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[1]Non-diagonal K-matrices exist but they are beyond the scope of this paper.

Now it can be shown that Hamiltonians corresponding to all nine boundary conditions are related to the second derivative of the boundary transfer matrix $\tau^{(a,b)}(u)$ (up to an unimportant additive constant)

$$
\begin{aligned}
H &= 2g\; H^{(a,b)}, \\
H^{(a,b)} &= \frac{\tau^{(a,b)''}(0)}{4(V+2W)} = \sum_{j=1}^{L-1} H^R_{j,j+1} + \frac{1}{2}\,\overset{1}{K}{}^{b'}_{-}(0) + \frac{1}{2(V+2W)}\Bigg[str_0\left(\overset{0}{K}{}^{a}_{+}(0)G_{L0}\right) \\
&\quad +2\,str_0\left(\overset{0}{K}{}^{a'}_{+}(0)H^R_{L0}\right) + str_0\left(\overset{0}{K}{}^{a}_{+}(0)\left(H^R_{L0}\right)^2\right)\Bigg],
\end{aligned}
\tag{III.10}
$$

where

$$
\begin{aligned}
V &= str_0 K^{a'}_{+}(0), \quad W = str_0\left(\overset{0}{K}{}^{a}_{+}(0)H^R_{L0}\right), \\
H^R_{j,j+1} &= P_{j,j+1}R'_{j,j+1}(0), \quad G_{j,j+1} = P_{j,j+1}R''_{j,j+1}(0)
\end{aligned}
\tag{III.11}
$$

with $P_{j,j+1}$ denoting the graded permutation operator acting on the j-th and $j+1$-th quantum spaces. More precisely, Case (i), Case (ii) and Case (iii) correspond to $H^{(I,I)}$, $H^{(II,II)}$ and $H^{(III,III)}$, respectively. We arrange the remaing six cases in the following order so that we have the correspondence: Case (iv) $\leftrightarrow H^{(I,II)}$, Case (v) $\leftrightarrow H^{(II,I)}$, Case (vi) $\leftrightarrow H^{(I,III)}$, Case (vii) $\leftrightarrow H^{(III,I)}$, Case (viii) $\leftrightarrow H^{(II,III)}$ and Case (ix) $\leftrightarrow H^{(III,II)}$.

Let us also remark that for general boundary parameters $\xi^a_\pm$ the boundary terms listed above break the original $gl(3|1)$ symmetry of the bulk model into $U(1)\times U(1)\times U(1)$ symmetry (generated by fermion number operators).

IV. BETHE ANSATZ SOLUTIONS

Having established the quantum integrability of the boundary model, we now solve it by using the coordinate space Bethe ansatz method. Following [11,9,3,10], we assume that the eigenfunction of Hamiltonian (II.2) takes the form

$$
\begin{aligned}
|\Psi\rangle &= \sum_{\{(x_j,\alpha_j)\}} \Psi_{\alpha_1,\cdots,\alpha_N}(x_1,\cdots,x_N)c^\dagger_{x_1\alpha_1}\cdots c^\dagger_{x_N\alpha_N}|0\rangle, \\
\Psi_{\alpha_1,\cdots,\alpha_N}(x_1,\cdots,x_N) &= \sum_P \epsilon_P A_{\alpha_{Q1},\cdots,\alpha_{QN}}(k_{PQ1},\cdots,k_{PQN})\exp(i\sum_{j=1}^N k_{P_j}x_j),
\end{aligned}
\tag{IV.1}
$$

where the summation is taken over all permutations and negations of $k_1,\cdots,k_N$, and Q is the permutation of the N particles such that $1\le x_{Q1}\le\cdots\le x_{QN}\le L$. The symbol ϵ_P is a sign factor ±1 and changes its sign under each 'mutation'. Substituting the wavefunction into the eigenvalue equation $H|\Psi\rangle = E|\Psi\rangle$, one gets

$$
\begin{aligned}
A_{\cdots,\alpha_j,\alpha_i,\cdots}(\cdots,k_j,k_i,\cdots) &= S_{ij}(k_i,k_j)A_{\cdots,\alpha_i,\alpha_j,\cdots}(\cdots,k_i,k_j,\cdots), \\
A_{\alpha_i,\cdots}(-k_j,\cdots) &= s^L(k_j;p_{1\alpha_i})A_{\alpha_i,\cdots}(k_j,\cdots), \\
A_{\cdots,\alpha_i}(\cdots,-k_j) &= s^R(k_j;p_{L\alpha_i})A_{\cdots,\alpha_i}(\cdots,k_j),
\end{aligned}
\tag{IV.2}
$$

where $S_{ij}(k_i,k_j)$ are the two-particle scattering matrices,

$$
S_{ij}(k_i,k_j) = \frac{\theta(k_i)-\theta(k_j)+icP_{ij}}{\theta(k_i)-\theta(k_j)+ic}
\tag{IV.3}
$$

where P_{ij} denotes the operator interchanging the species variables α_i and α_j, $(\alpha_i,\alpha_j=+,0,-)$,the rapidities $\theta(k_j)$ are related to the single-particle quasi-momenta k_j by $\theta(k)=\frac{1}{2}\tan(\frac{k}{2})$ and the dependence on the system parameter g is incorporated in the parameter $c=1/g$. $s^L(k_j;p_{1\alpha_i})$ and $s^R(k_j;p_{L\alpha_i})$ are the boundary scattering matrices,

$$
\begin{aligned}
s^L(k_j;p_{1\alpha_i}) &= \frac{1-p_{1\alpha_i}e^{ik_j}}{1-p_{1\alpha_i}e^{-ik_j}}, \\
s^R(k_j;p_{L\alpha_i}) &= \frac{1-p_{L\alpha_i}e^{-ik_j}}{1-p_{L\alpha_i}e^{ik_j}}e^{2ik_j(L+1)}
\end{aligned}
\tag{IV.4}
$$

with $p_{1\alpha_i}$ and $p_{L\alpha_i}$ being given by the following formulae, corresponding to (II.4 – II.6), respectively,

$$\begin{aligned}\text{Case (i)}:\ & p_{1+}=p_{10}=p_{1-}\equiv p_1=-1-\frac{2g}{2-\xi_-^I},\\ & p_{L+}=p_{L0}=p_{L-}\equiv p_L=-1-\frac{2g}{2-\xi_+^I};\end{aligned}\tag{IV.5}$$

$$\begin{aligned}\text{Case (ii)}:\ & p_{1+}=-1,\quad p_{10}=p_{1-}=-1+\frac{2g}{\xi_-^{II}},\\ & p_{L+}=-1,\quad p_{L0}=p_{L-}=-1+\frac{2g}{\xi_+^{II}};\end{aligned}\tag{IV.6}$$

$$\begin{aligned}\text{Case (iii)}:\ & p_{1+}=p_{10}=-1,\quad p_{1-}=-1+\frac{2g}{\xi_-^{III}},\\ & p_{L+}=p_{L0}=-1,\quad p_{L-}=-1+\frac{2g}{\xi_+^{III}}.\end{aligned}\tag{IV.7}$$

As is seen above, the two-particle S-matrix (IV.3) is nothing but the R-matrix of the $gl(3)$-invariant Heisenberg isotropic magnetic chain and thus satisfies the quantum Yang-Baxter equation (QYBE),

$$S_{ij}(k_i,k_j)S_{il}(k_i,k_l)S_{jl}(k_j,k_l)=S_{jl}(k_j,k_l)S_{il}(k_i,k_l)S_{ij}(k_i,k_j).\tag{IV.8}$$

It can be checked that the boundary scattering matrices s^L and s^R obey the reflection equations:

$$\begin{aligned}&S_{ji}(-k_j,-k_i)s^L(k_j;p_{1\alpha_j})S_{ij}(-k_i,k_j)s^L(k_i;p_{1\alpha_i})\\ &\qquad=s^L(k_i;p_{1\alpha_i})S_{ji}(-k_j,k_i)s^L(k_j;p_{1\alpha_i})S_{ij}(k_i,k_j),\\ &S_{ji}(-k_j,-k_i)s^R(k_j;p_{L\alpha_j})S_{ij}(k_i,-k_j)s^R(k_i;p_{L\alpha_i})\\ &\qquad=s^R(k_i;p_{L\alpha_i})S_{ji}(k_j,-k_i)s^R(k_j;p_{L\alpha_i});p_{\alpha_i})S_{ji}(k_j,k_i).\end{aligned}\tag{IV.9}$$

This is seen as follows. One introduces the notation

$$s(k;p)=\frac{1-pe^{-ik}}{1-pe^{ik}}.\tag{IV.10}$$

Then the boundary scattering matrices $s^L(k_j;p_{1\alpha_i}),\ \ s^R(k_j;p_{L\alpha_i})$ can be written as, corresponding to (IV.5 – IV.7), respectively,

$$\begin{aligned}\text{Case i}:\ & s^L(k_j;p_{1\alpha_i})=s(-k_j;p_1)I,\\ & s^R(k_j;p_{L\alpha_i})=e^{ik_j2(L+1)}s(k_j;p_L)I;\end{aligned}\tag{IV.11}$$

$$\begin{aligned}\text{Case ii}:\ & s^L(k_j;p_{1\alpha_i})=s(-k_j;p_{1+})\begin{pmatrix}1&0&0\\0&\frac{\zeta_-+\lambda_j}{\zeta_--\lambda_j}&0\\0&0&\frac{\zeta_-+\lambda_j}{\zeta_--\lambda_j}\end{pmatrix},\\ & s^R(k_j;p_{L\alpha_i})=e^{ik_j2(L+1)}s(k_j;p_{L+})\begin{pmatrix}1&0&0\\0&\frac{\zeta_+-\lambda_j}{\zeta_++\lambda_j}&0\\0&0&\frac{\zeta_+-\lambda_j}{\zeta_++\lambda_j}\end{pmatrix};\end{aligned}\tag{IV.12}$$

$$\begin{aligned}\text{Case iii}:\ & s^L(k_j;p_{1\alpha_i})=s(-k_j;p_{1+})\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{\kappa_-+\lambda_j}{\kappa_--\lambda_j}\end{pmatrix},\\ & s^R(k_j;p_{L\alpha_i})=e^{ik_j2(L+1)}s(k_j;p_{L+})\begin{pmatrix}1&0&0\\0&1&0\\0&0&\frac{\kappa_+-\lambda_j}{\kappa_++\lambda_j}\end{pmatrix},\end{aligned}\tag{IV.13}$$

where I stands for 3×3 identity matrix and $p_{1+},\ p_{L+}$ are the ones given in (IV.6); $\zeta_\pm,\ \kappa_\pm$ are parameters defined by

$$\zeta_\pm=\frac{i(g-\xi_\pm^{II})}{2g},\qquad \kappa_\pm=\frac{i(g-\xi_\pm^{III})}{2g}.\tag{IV.14}$$

The boundary scattering matrices for cases (iv) – (ix) can be easily built from (IV.11), (IV.12) and (IV.13). We immediately see that (IV.11) are the trivial solutions of the reflection equations (IV.9), whereas (IV.12) and (IV.13) are the diagonal solutions [1,2].

The diagonalization of Hamiltonian (II.2) reduces to solving the following matrix eigenvalue equation

$$T_j t = t, \qquad j = 1, \cdots, N, \tag{IV.15}$$

where t denotes an eigenvector on the space of the spin variables and T_j takes the form

$$T_j = S_j^-(k_j) s^L(-k_j; p_{1\sigma_j}) R_j^-(k_j) R_j^+(k_j) s^R(k_j; p_{L\sigma_j}) S_j^+(k_j) \tag{IV.16}$$

with

$$\begin{aligned} S_j^+(k_j) &= S_{j,N}(k_j, k_N) \cdots S_{j,j+1}(k_j, k_{j+1}), \\ S_j^-(k_j) &= S_{j,j-1}(k_j, k_{j-1}) \cdots S_{j,1}(k_j, k_1), \\ R_j^-(k_j) &= S_{1,j}(k_1, -k_j) \cdots S_{j-1,j}(k_{j-1}, -k_j), \\ R_j^+(k_j) &= S_{j+1,j}(k_{j+1}, -k_j) \cdots S_{N,j}(k_N, -k_j). \end{aligned} \tag{IV.17}$$

This problem can be solved using the algebraic Bethe ansatz method. The Bethe ansatz equations are

$$\begin{aligned} e^{ik_j 2(L+1)} F(k_j; p_{1+}, p_{L+}) &= \prod_{\sigma=1}^{M_1} \frac{\theta_j - \lambda_\sigma^{(1)} + ic/2}{\theta_j - \lambda_\sigma^{(1)} - ic/2} \frac{\theta_j + \lambda_\sigma^{(1)} + ic/2}{\theta_j + \lambda_\sigma^{(1)} - ic/2}, \\ \prod_{j=1}^{N} \frac{\lambda_\sigma^{(1)} - \theta_j + ic/2}{\lambda_\sigma^{(1)} - \theta_j - ic/2} \frac{\lambda_\sigma^{(1)} + \theta_j + ic/2}{\lambda_\sigma^{(1)} + \theta_j - ic/2} &= G(\lambda_\sigma^{(1)}; \zeta_-, \zeta_+) \prod_{\substack{\rho=1 \\ \rho\neq\sigma}}^{M_1} \frac{\lambda_\sigma^{(1)} - \lambda_\rho^{(1)} + ic}{\lambda_\sigma^{(1)} - \lambda_\rho^{(1)} - ic} \frac{\lambda_\sigma^{(1)} + \lambda_\rho^{(1)} + ic}{\lambda_\sigma^{(1)} + \lambda_\rho^{(1)} - ic} \\ &\quad \prod_{\rho=1}^{M_2} \frac{\lambda_\sigma^{(1)} - \lambda_\rho^{(2)} - ic/2}{\lambda_\sigma^{(1)} - \lambda_\rho^{(2)} + ic/2} \frac{\lambda_\sigma^{(1)} + \lambda_\rho^{(2)} - ic/2}{\lambda_\sigma^{(1)} + \lambda_\rho^{(2)} + ic/2}, \\ &\quad \sigma = 1, \cdots, M_1, \\ \prod_{\rho=1}^{M_1} \frac{\lambda_\gamma^{(2)} - \lambda_\rho^{(1)} + ic/2}{\lambda_\gamma^{(2)} - \lambda_\rho^{(1)} - ic/2} \frac{\lambda_\gamma^{(2)} + \lambda_\rho^{(1)} + ic/2}{\lambda_\gamma^{(2)} + \lambda_\rho^{(1)} - ic/2} &= K(\lambda_\gamma^{(2)}; \kappa_-, \kappa_+) \prod_{\substack{\rho=1 \\ \rho\neq\gamma}}^{M_2} \frac{\lambda_\gamma^{(2)} - \lambda_\rho^{(2)} + ic}{\lambda_\gamma^{(2)} - \lambda_\rho^{(2)} - ic} \frac{\lambda_\gamma^{(2)} + \lambda_\rho^{(2)} + ic}{\lambda_\gamma^{(2)} + \lambda_\rho^{(2)} - ic}, \\ &\quad \gamma = 1, \cdots, M_2, \end{aligned} \tag{IV.18}$$

where

$$F(k_j; p_{1+}, p_{L+}) = s(k_j; p_{1+}) s(k_j; p_{L+}), \text{(for all cases)}$$

$$G(\lambda_\sigma^{(1)}; \zeta_-, \zeta_+) = \begin{cases} 1 & case\ (i) \\ \frac{(\zeta_- + \lambda_\sigma^{(1)} + \frac{ic}{2})}{(\zeta_- - \lambda_\sigma^{(1)} + \frac{ic}{2})} \frac{(\zeta_+ + \lambda_\sigma^{(1)} + \frac{ic}{2})}{(\zeta_+ - \lambda_\sigma^{(1)} + \frac{ic}{2})} & case\ (ii) \\ 1 & case\ (iii) \\ \frac{(\zeta_+ + \lambda_\sigma^{(1)} + \frac{ic}{2})}{(\zeta_+ - \lambda_\sigma^{(1)} + \frac{ic}{2})} & case\ (iv) \\ \frac{(\zeta_- + \lambda_\sigma^{(1)} + \frac{ic}{2})}{(\zeta_- - \lambda_\sigma^{(1)} + \frac{ic}{2})} & case\ (v) \\ 1 & case\ (vi) \\ 1 & case\ (vii) \\ \frac{(\zeta_- + \lambda_\sigma^{(1)} + \frac{ic}{2})}{(\zeta_- - \lambda_\sigma^{(1)} + \frac{ic}{2})} & case\ (viii) \\ \frac{(\zeta_+ + \lambda_\sigma^{(1)} + \frac{ic}{2})}{(\zeta_+ - \lambda_\sigma^{(1)} + \frac{ic}{2})} & case\ (ix) \end{cases} .$$

$$K(\lambda_\gamma^{(2)};\kappa_-,\kappa_+)=\begin{cases}1 & case\ (i)\\ 1 & case\ (ii)\\ \frac{(\kappa_-+\lambda_\gamma^{(2)}+ic)}{(\kappa_--\lambda_\gamma^{(2)}+ic)}\frac{(\kappa_++\lambda_\gamma^{(2)}+ic)}{(\kappa_+-\lambda_\gamma^{(2)}+ic)} & case\ (iii)\\ 1 & case\ (iv)\\ 1 & case\ (v)\\ \frac{(\kappa_++\lambda_\gamma^{(2)}+ic)}{(\kappa_+-\lambda_\gamma^{(2)}+ic)} & case\ (vi)\\ \frac{(\kappa_-+\lambda_\gamma^{(2)}+ic)}{(\kappa_--\lambda_\gamma^{(2)}+ic)} & case\ (vii)\\ \frac{(\kappa_++\lambda_\gamma^{(2)}+ic)}{(\kappa_+-\lambda_\gamma^{(2)}+ic)} & case\ (viii)\\ \frac{(\kappa_-+\lambda_\gamma^{(2)}+ic)}{(\kappa_--\lambda_\gamma^{(2)}+ic)} & case\ (ix)\end{cases}. \tag{IV.19}$$

The energy eigenvalue E of the model is given by $E=-2\sum_{j=1}^{N}\cos k_j$ (modular an unimportant additive constant coming from the chemical potential term).

V. CONCLUSION

In conclusion, we have studied integrable open-boundary conditions for the eight-state supersymmetric U model. The quantum integrability of the system follows from the fact that the Hamiltonian may be embedded into a one-parameter family of commuting transfer matrices. Moreover, the Bethe ansatz equations are derived by use of the coordinate space Bethe ansatz approach. This provides us with a basis for computing the finite size corrections (see, e.g. [11]) to the low-lying energies in the system, which in turn allow us to use the boundary conformal field theory technique to study the critical properties of the boundary. The details will be treated in a separate publication.

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